

# Infrared Instability in Graviton Higgs Theory

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## Abstract

Gravitons minimally coupled to Higgs fields in a background Minkowski space-time is shown to develop an instability in their propagators in presence of a spacetime-independent Higgs field background, due to the appearance of a tachyonic pole. The one loop effective potential (for constant Higgs field backgrounds) is shown to develop an infrared instability in the form of acquiring an imaginary part, which can be traced to the tachyonic pole in the graviton propagator. This instability is analogous to the finite temperature infrared instability of a gas of gravitons coupled to fermions found by Gross et. al., even though it already exists at zero temperature; it is thus reminiscent of the Jeans instability thought to be at the heart of structure formation in the early universe. A finite temperature analysis of the effective potential at one loop shows that in the high temperature limit, the zero-temperature instability is in fact *reinforced* by finite temperature effects. In the low temperature limit, the finite temperature contribution to the imaginary part of the effective potential exhibits a damped oscillatory behaviour; all thermal effects are damped out as the temperature vanishes, consistent with the zero-temperature result.

## 1 Introduction

Stability of flat spacetime under quantum gravitational fluctuations has been studied extensively since the incipient work on the Euclidean path integral formulation of gravity [1, 2]. Employing a saddle-point approximation in the Euclidean partition function, Gross et. al. [3] show that flat space is stable at zero temperature both classically and quantum mechanically under perturbative quantum fluctuations of Euclidean 4-space. However, when the system is kept in contact with a heat bath, the self-gravitating system becomes unstable, both by itself (vacuum) and in the presence of massless spinor fields. This is unlike in the case of an electrical plasma where charge carriers produce a screening effect over the fluid (Debye screening). This distinct feature of gravity is the source of several instabilities. In classical Newtonian gravity one such instability occurs when we treat the universe as being filled with a static, homogeneous nonrelativistic fluid. For long-wavelength gravitational perturbations the system develops an instability. This instability is very often be related to classical Jeans instability [4]. Jeans' Universe is filled with a non

viscous fluid with mass density  $\rho$ , pressure  $p$ , and velocity  $\vec{v}$  satisfying the usual continuity and Euler equation. To analyze the dynamics of the system one considers perturbations  $\rho_1, p_1, v_1, g_1$  around an equilibrium configuration which is taken to be a uniform static fluid. The effect of gravitation are also ignored in the unperturbed solution. When we solve for the density perturbation it takes the usual plane waveform

$$\rho_1 \propto \exp\{i\mathbf{k} \cdot \mathbf{x} - i\omega t\}$$

leading to a dispersion relation,  $\omega^2 = v_s^2 \mathbf{k}^2 - 4\pi G \rho$  where  $\omega$  is imaginary for wave numbers below the critical value

$$k_J = \left( \frac{4\pi G \rho}{v_s^2} \right)^{\frac{1}{2}}$$

So the perturbation  $\rho_1$  has a runaway mode below this value and can result in an exponential growth or decay of the disturbances [5].

In a classic work, Gross et. al. [3] consider a gas of gravitons in thermal contact at finite spatial volume and interacting with thermally excited fermions. Integrating over the fermionic degrees of freedom, the graviton is shown to acquire an imaginary mass leading to a tachyonic instability. The presence of a heat bath as a source for inducing thermal fluctuations is crucial in this work, as is evident from the fact that the induced masses have power law dependence on the temperature. Indeed, we reiterate that no instability is reported in ref. [3] at zero temperature.

In this paper we show that even at zero temperature, when gravitons couple to massless scalar field backgrounds which are spacetime independent, a similar instability appears, with the effective one-loop graviton propagator acquiring a tachyonic pole. This, in turn, leads to the appearance of an imaginary contribution in the one-loop effective action for a wide class of theories involving graviton and scalars, when evaluated using the Euclidean path integral saturated at a saddle point characterized by a flat Euclidean metric and a constant scalar background. This implies that a graviton fluctuations coupled to constant scalar field background at  $T = 0$  in flat spacetime plays a role similar to gravitons in a finite temperature heat-bath in inducing an instability in flat spacetime. It is perhaps not inappropriate to state that this phenomenon has been an issue not particularly well-understood [6, 7, 8] as to how the instability resolves itself. It is not unlikely that the instability will involve decay to a de Sitter spacetime, but the actual proof of this is beyond this note.

With the result of ref. [3], the issue now is : what happens when the graviton-Higgs system is made to interact with a heat bath ? The thermal contribution of the one loop effective potential is thus important to investigate, to ascertain whether the zero temperature instability is reinforced or weakened. Doubtlessly, the result of this assay will have implications for inflation and perhaps also for the electroweak phase transition in the early universe. The recent discovery of a 125 GeV scalar boson at CERN lends special credence to theories with gravitons interacting with Higgs fields vis-a-vis their implication for various instabilities in the early universe [9, 10]. In this paper we explore the interplay of thermal vs Higgs/graviton induced instability in the finite temperature through computation of the one loop graviton-Higgs effective potential. It is found that the effect of constant scalar background is being amplified in the high temperature limit

of Higgs-graviton effective potential. The low temperature limit, on the other hand, shows a rather interesting behaviour : in the physically relevant region the temperature dependent imaginary part oscillates with a damping amplitude. This oscillation may be a reminiscence of the instability of flat background under perturbation in presence of interaction between gravitons and thermally excited matter fields.

The plan of the paper as follows : In the next section we show the appearance of the tachyonic pole in the classical graviton propagator in minimally-coupled graviton-Higgs theory. The following section (section 3) deals with the one-loop effective potential of the graviton-Higgs theory, where the instability is exhibited, and its presence traced to the tachyonic pole in the graviton propagator. There is also a comparison of the nature of the one-loop effect between gauge-Higgs and graviton-Higgs theory. This is followed in section 4 by a detailed discussion on the one-loop effective potential at finite temperature. The infrared limit describes an interesting situation exhibiting an instability due to the temperature dependent contribution to the effective potential developing an additional imaginary part over and above the one in the  $T = 0$  limit. Various aspects of this instability are discussed. We conclude in section 5 with a critical look at our results and a future outlook.

## 2 Tachyonic mode at zero temperature propagator

In ref. [3] it is shown that flat space is stable both quantum mechanically and classically under small perturbations due to gravitons and spinor fields at zero temperature. They also showed that when a gas of gravitons is kept at finite temperature, an instability, stemming from thermally generated graviton modes, appears. This induces a Jeans-like instability since the thermally excited modes interact with gravitons. They also show in the same paper that for a theory with gravitons coupled to thermally excited fermions, the one-loop graviton propagator contains a tachyonic term which can be interpreted as a mass term for the *longitudinal* mode of the graviton  $h_{00}$ ; this mass is of magnitude

$$m_g^2 = -14/15\pi^3 GT^4 \quad (1)$$

The generation of an imaginary mass term when gravitons couple with thermally excited matter field is a generic feature. This also holds for the case of scalar fields at finite temperature. In fact the mass induced for the case of scalar field can again be traced from the self energy component  $\Pi_{00,00}$ . The longitudinal part of graviton  $h_{00}$  here again develops a mass term due to thermal fluctuations [11]. The value in this case is

$$m_g^2 = -\frac{4}{5}\pi^3 GT^4 \quad (2)$$

All these effects are purely thermal, implying that flat space is unstable.

We now go back to the case without an external heat bath, and consider a minimally coupled graviton-Higgs theory at  $T = 0$ . Assuming that the theory has a vacuum characterised by a constant value of the Higgs field which plays the role of an external background for the gravitons, an instability similar to the finite temperature case is discerned. In other words, the scalar field background plays the role of a heat bath which induces a tachyonic instability in the graviton modes even at zero temperature.

To evaluate the effective propagator for Higgs-graviton theory we start from the Lagrangian for Einstein General Relativity and expand it around the flat space i.e. we employ weak field approximation.

$$\sqrt{-g}\mathcal{L}_g = \sqrt{-g} \left[ \frac{1}{\kappa^2} \mathcal{R} \right] \quad (3)$$

where  $\kappa^2 = 16\pi G$  ;  $g = \det g_{\mu\nu}$  and  $R = g^{\mu\nu} \mathcal{R}_{\mu\nu}$

The Lagrangian for Gravity coupled to a massless scalar field,

$$\sqrt{-g}\mathcal{L} = \frac{1}{\kappa^2} R + \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \sqrt{-g} V(\phi) \quad (4)$$

Expanding the metric around a flat background we get,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (5)$$

where the fluctuations  $h_{\mu\nu}$  are small,  $|h_{\mu\nu}| < 1$ . For the decomposition (5), the inverse of the metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu{}_\lambda h^{\lambda\nu} + \dots \quad (6)$$

Furthermore, the determinant of the metric, which will be needed in the following, will be given by:

$$(-g)^{\frac{1}{2}} = 1 + \kappa \frac{1}{2} h^\alpha{}_\alpha - \kappa^2 \frac{1}{4} h^\alpha{}_\beta h^\beta{}_\alpha + \kappa^2 \frac{1}{8} (h^\alpha{}_\alpha)^2 + \dots \quad (7)$$

The quadratic part of the Lagrangian from pure gravity sector is given by

$$\mathcal{L}_g = -\frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{2} \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\mu h^\mu{}_\nu \partial_\alpha h^{\nu\alpha} + \frac{1}{4} \partial_\mu h \partial^\mu h. \quad (8)$$

This Lagrangian is invariant under linear gauge transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (9)$$

The matter sector is also expanded around a space-time constant background  $\phi = \phi_0 + \Phi$

$$\sqrt{-g}\mathcal{L}_m = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{\kappa}{2} h V'(\phi_0) \Phi + \kappa^2 \left( \frac{1}{4} h^\alpha{}_\beta h^\beta{}_\alpha - \frac{1}{8} h^2 \right) V(\phi_0) \quad (10)$$

If we write down an effective linearized equation of motion for the graviton field from the quadratic part of the Lagrangian, we get an equation

$$\mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta} = \kappa T^{\mu\nu}, \quad (11)$$

where  $T^{\mu\nu}$  contains bilinear interaction terms containing appropriately contracted products of terms *linear* in the scalar and graviton fluctuation fields. The operator  $\mathcal{O}^{\mu\nu,\alpha\beta}$  can

be extracted from the bilinear effective Lagrangian (in Minkowski metric). We write it here in Fourier space.

$$\mathcal{O}^{\mu\nu,\alpha\beta} = \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}(-k^2 + \kappa^2 V) - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}(-k^2 + \frac{\kappa^2 V}{2}) - \eta^{\mu\nu}k^\alpha k^\beta + \eta^{\mu\alpha}k^\nu k^\beta \quad (12)$$

This operator is perfectly invertible and we can compute the graviton propagator from this without any gauge fixing. This is clearly due to our separation of the various graviton terms in the (bi-)linearized Lagrangian into a part which contribute exclusively to the ‘kinetic energy’ operator  $\mathcal{O}^{\mu\nu,\alpha\beta}$ , and those which contribute to the ‘source’ energy-momentum tensor  $T^{\mu\nu}$  in (11). In effect, the operator  $\mathcal{O}^{\mu\nu,\alpha\beta}$  is already gauge-fixed.

The graviton propagator is thus given by

$$\begin{aligned} -iD_{\mu\nu,\alpha\beta}(k) &= -\frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}}{k^2 - \kappa^2 V} + \frac{\eta_{\mu\nu}\eta_{\alpha\beta}(2k^4 + \kappa^2 K^2 V - \kappa^4 V^2)}{(k^2 - \kappa^2 V)(3k^4 + \frac{3}{2}\kappa^2 k^2 V - \kappa^4 V^2)} \\ &- \frac{(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu)\kappa^2 V}{2(k^2 - \kappa^2 V)(3k^4 + \frac{3}{2}\kappa^2 k^2 V - \kappa^4 V^2)} \\ &+ \frac{(\eta_{\mu\alpha}k_\nu k_\beta + \eta_{\mu\beta}k_\nu k_\alpha + \eta_{\nu\alpha}k_\mu k_\beta + \eta_{\nu\beta}k_\mu k_\alpha)(3k^2 - \kappa^2 V)\kappa^2 V}{4(k^2 - \frac{\kappa^2 V}{2})(k^2 - \kappa^2 V)(3k^4 + \frac{3}{2}\kappa^2 k^2 V - \kappa^4 V^2)} \quad (13) \end{aligned}$$

This propagator contains complex poles as can be seen from the denominator of second, third and last terms of (13). The factor  $3k^4 + \frac{3}{2}\kappa^2 k^2 V - \kappa^4 V^2$  has a complex pole at

$$k^2 = -\left(\frac{1}{4} + \frac{\sqrt{57}}{2}\right)\kappa^2 V$$

In the infrared limit there will be a tachyonic mode due to this pole at the frequency  $k_0 = \pm\sqrt{\mathbf{k}^2 - \kappa^2 V(\phi_0)}$ . Recall that the dispersion relation in Jeans’ treatment of the gravitational instability of a homogeneous fluid is  $\omega^2 = v_s^2 \mathbf{k}^2 - 4\pi G\rho$ . Hence

$$h_{\mu\nu}(k) = D_{\mu\nu,\alpha\beta}(k) T^{\alpha\beta}(k)$$

will produce a runaway solution triggering a Jeans-like instability.

Thus in the infrared limit the constant scalar background induces an imaginary mass proportional to the potential of the field. If we choose  $V(\phi)$  to be  $\lambda\frac{\phi^4}{4!}$  then the induced mass is proportional to the fourth power of constant background  $\phi_0$ . It is perhaps not a coincidence that in (2) the induced tachyonic mass is proportional to the fourth power of the *temperature*.

We now proceed to investigate the effect of this graviton propagator in the one-loop effective potential.

### 3 One-loop Effective potential for graviton-Higgs theory

In the path integral formulation of quantum field theory, the effective potential (the sum of all one particle irreducible vacuum Feynman graphs with zero external momenta)

is a useful tool to investigate perturbatively induced symmetry breaking in the theory (the Coleman Weinberg mechanism) [16]. The effective potential enables us to survey all possible vacuum states of the quantum theory at once, and to compute higher order corrections before identifying the physical (perturbative) vacuum state of the theory.

Here, we calculate the one-loop effective potential for a theory where gravity is coupled to a Higgs field minimally. In the one-loop approximation,

$$\Gamma^{(1)}[\Phi] = S[\Phi] - \frac{1}{2} \text{Tr} \ln S_2[\Phi]$$

The one-loop effective potential is obtained from one-loop effective *action* by setting the mean field a constant value  $\Phi(x) = \phi_0 = \text{constant}$

$$V_{eff}^{(1)} = V(\phi_0) + \frac{1}{2} \text{Tr} \ln S_2[\Phi]|_{\Phi=\phi_0}$$

When the source is switched off then

$$\frac{\partial V_{eff}}{\partial \Phi}|_{\Phi=\phi_0} = 0$$

gives the minimum of the effective action. If this equation is satisfied by some non-zero  $\phi_0$ , this may be the origin of a vacuum instability.

The full Lagrangian to quadratic order in fluctuating fields is given by,

$$\mathcal{L}_q = \{ \mathcal{L}_m^{(0)} + \mathcal{L}_g^{(2)} + \mathcal{L}_{gf} + \mathcal{L}_{ghost} + \mathcal{L}_m^{(2)} \} \quad (14)$$

The gauge fixing and ghost Lagrangians are

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha} \left[ \partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h \right]^2 \quad (15)$$

and

$$\mathcal{L}_{ghost} = \frac{1}{2} \partial_\alpha \zeta_\mu \partial^\alpha \bar{\zeta}^\mu \quad (16)$$

Retaining terms upto quadratic in fluctuating field  $\Phi$  we get the relevant Lagrangian for one-loop EP

$$\begin{aligned} \mathcal{L}_{rel} &= \frac{1}{2} \Phi (-\square_E + V'') \Phi + \frac{1}{4} h_{\mu\nu} (-\square_E - \kappa^2 V) h^{\mu\nu} \\ &- \frac{1}{4} h \left[ \left(1 - \frac{1}{2\alpha}\right) (-\square_E) - \frac{\kappa^2 V}{2} \right] h \\ &+ \frac{1}{2} h_{\mu\nu} \left(1 - \frac{1}{\alpha}\right) \partial^\nu \partial_\rho h^{\mu\rho} - h \left(1 - \frac{1}{\alpha}\right) \partial_\mu \partial_\nu h^{\mu\nu} \end{aligned} \quad (17)$$

This can also be seen if we calculate the effective propagator from the bilinear action of this theory after expanding around a constant background. As we can easily verify, the full Lagrangian i.e. the sum of  $L_g$  and  $L_m$  after we expand around the backgrounds does

not have the gauge invariance (9) anymore. This is because the full Lagrangian does not possess the gauge invariance for the fluctuating degrees of freedom only.

The Lagrangian (3) can be written in a compact form following [17],

$$\mathcal{L}_{rel} = \frac{1}{2}\Phi(-\square_E + V'')\Phi + \frac{1}{2}\Psi_i M_{ij} \Psi_j$$

where  $\Psi_i$  ( $i = 1, 2, \dots, 10$ ) represents ten independent components of  $h_{\mu\nu}$ .

The eigenvalues for the matrix  $M$  in  $\alpha = 1$  gauge are,

$$\lambda_i = k^2 - \kappa^2 V ; (1 \leq i \leq 6) \quad (18)$$

$$\lambda_i = \frac{1}{2}(k^2 - \kappa^2 V) ; (7 \leq i \leq 9) \quad (19)$$

$$\lambda_{10} = -\frac{1}{2} \left[ \frac{(k^2 - \kappa^2 V)(k^2 + V'') + 2\kappa^2 V'^2}{k^2 + V''} \right] \quad (20)$$

. The eigenvalue for the operator coming from the quadratic part of the scalar field is given by,

$$\lambda_\phi = k^2 + V'' \quad (21)$$

The effective potential in terms of the momentum integrals is given by,

$$V_{eff} = V + \frac{9}{2} \int \frac{d^4 k}{(2\pi)^4} \ln(k^2 - \kappa^2 V) + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left[ k^4 + (V'' - \kappa^2 V)k^2 + \kappa^2(2V'^2 - VV'') \right] \quad (22)$$

Evaluating the momentum integrals with a cut-off  $\Lambda$  the one loop effective potential is given by,

$$\begin{aligned} V_{eff}(\phi_0) = & V + \frac{9}{32\pi^2} \left[ \frac{\kappa^4 V^2}{2} \left( \ln \frac{\kappa^2 V}{\Lambda^2} - \frac{1}{2} \right) - \kappa^2 V \Lambda^2 \right] - \frac{i\kappa^4 V^2}{2\pi} \\ & + \frac{1}{32\pi^2} \left[ (V'' - \kappa^2 V)\Lambda^2 + \frac{a^2 - 2b}{4} \left( \ln \frac{b}{\Lambda^4} - 1 \right) \right] \\ & + \frac{a\sqrt{a^2 - 2b}}{64\pi^2} \ln \left[ \frac{a + \sqrt{a^2 - 4b}}{a - \sqrt{a^2 - 4b}} \right] \end{aligned} \quad (23)$$

where

$$V = \frac{\lambda \phi_0^4}{4!}$$

and

$$\begin{aligned} a &= V'' - \kappa^2 V \\ b &= \kappa^2(2V'^2 - VV'') \end{aligned}$$

The source of this imaginary part is the infrared sector of the graviton scalar mode-Higgs interaction : the relative sign of the graviton-Higgs couplings actually gives rise to this. In the infrared limit the functional traces become non-analytic due to negative logarithms. Similar results have been obtained in refs [6, 7] etc. In a related work, Fradkin et al [14] have shown that for a gauged supergravity theory one of the modes in the spectral decomposed one-loop

operators in DeSitter background contains negative modes. In some higher derivative gravity theories, with non-minimal coupling to scalar fields, similar imaginary terms in the effective potential have also been observed [13, 12].

An interesting feature of the one-loop effective potential is that the effect completely disappears if the classical Higgs potential is set to zero. This is in contrast to the flat spacetime gauge field theories where a minimally coupled Higgs field *generates* an effective potential perturbatively, even if the classical potential vanishes. This is because in Higgs-graviton theory the absence of a classical Higgs potential, the Higgs field has no other coupling to the graviton field when expanded around a constant vacuum value. In standard electroweak theory in flat spacetime, in contrast, the classical Lagrangian has Higgs-gauge field seagull terms which lead to the one loop effective potential [15] even in the absence of a classical potential. This does not happen in perturbative quantum gravity since there is no such interaction for a constant Higgs background. In fact, this feature of scalar-graviton theory appears to persist to higher orders of perturbation theory for spacetime independent Higgs backgrounds.

## 4 Effect of Finite Temperature

The appearance of an imaginary part in the zero temperature one-loop effective potential prompts us to investigate the situation for the finite temperature counterpart of the theory. We have already cited the literature where a tachyonic pole in the one loop graviton self-energy has been discerned, leading to an instability in the theory. The issues addressed in this section are : (a) how this instability manifests in the one-loop effective potential, and (b) if there are additional *imaginary* temperature-dependent contributions at one loop, whether these contributions neutralize the zero-temperature imaginary part of the effective potential found in the last section, or *enhance* it.

From (23) we can write down the expression for the one-loop effective potential in momentum space in a slightly modified form,

$$V_{eff} = V + \frac{9}{2} \int \frac{d^4 k}{(2\pi)^4} \ln(k^2 - \kappa^2 V) + \sum_{i=1}^2 \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln[k^4 + A_i] \quad (24)$$

with  $A_i$ 's are root of the quartic equation  $k^4 + ak^2 + b = 0$  where  $a = V'' - \kappa^2 V$  and  $b = \kappa^2(2V'^2 - VV'')$ .

To obtain finite temperature effective potential we have to shift the momentum integrals of (24) by

$$\begin{aligned} \int d^4 k &\rightarrow T \sum_n \int d^3 \mathbf{k} \\ k &\rightarrow (2\pi nT, \mathbf{k}) \end{aligned}$$

Thus now, the finite temperature counterpart of Euclidean momentum integrals become [19, 18],

$$V_{eff} = V_0 + \frac{1}{2} T \sum_{i=1}^2 \sum_{n=-\infty}^{\infty} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(\mathbf{k}^2 + 4\pi^2 n^2 T^2 + A_i) + \frac{9}{2} T \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(\mathbf{k}^2 + 4\pi^2 n^2 T^2 - \kappa^2 V) \quad (25)$$



We can represent the above integrals in a general form,

$$I(t, u) = \frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(\mathbf{k}^2 + tn^2 + u) \quad (26)$$

Here  $t = 4\pi^2 T^2$ . Since  $I(t, u)$  is a divergent quantity we have to regularize this integral. Dimensional regularization is most convenient to evaluate such integrals. We perform an integral transform to tackle the infinite sum in the expression. The basic integral is,

$$I(t, u, d) = \frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(\mathbf{k}^2 + tn^2 + u) = -\frac{t^{\frac{1}{2}}}{2\pi} \sum_{n=-\infty}^{n=\infty} \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\tau \tau^{-d/2-1} e^{-\tau(tn^2+u)}, \quad (27)$$

where we have used the relation

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} \ln(\mathbf{k}^2 + tn^2 + u) = -\frac{\partial}{\partial \alpha} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{(\mathbf{k}^2 + tn^2 + u)^\alpha} \Big|_{\alpha=0} = -\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} (tn^2 + u)^{\frac{d}{2}}, \quad (28)$$

and also assumed that the  $\tau$  integration has no singularities. To evaluate the integral (27) we perform a large temperature expansion of the integrand. At high temperature limit i.e. for  $\frac{u}{t} \ll 1$  we can write the sum over  $n$  as a binomial expansion in  $\frac{u}{t}$

$$\sum_{n=1}^{\infty} (tn^2 + u)^{\frac{d}{2}} = \sum_{n=1}^{\infty} t^{\frac{d}{2}} \left[ n^d + \left(\frac{d}{2}\right) \left(\frac{u}{t}\right) \frac{1}{n^{2-d}} \right] \quad (29)$$

$$+ \frac{1}{2} \left(\frac{d}{2}\right) \left(\frac{d}{2} - 1\right) \left(\frac{u}{t}\right)^2 \frac{1}{n^{4-d}} + O\left(\frac{u}{t}\right)^3 \Big] \quad (30)$$

$$= t^{\frac{d}{2}} \left[ \zeta(-d) + \left(\frac{d}{2}\right) \zeta(2-d) \left(\frac{u}{t}\right) + \zeta(4-d) \left(\frac{d}{4}\right) \left(\frac{d}{2} - 1\right) + O\left(\frac{u}{t}\right)^3 \right] \quad (31)$$

where we have used the definition of Riemann zeta function

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n > 1$$

and its analytic continuation to the region  $n < 1$ . One can now easily extract the pole part of the integral. Defining  $\epsilon = 3 - d$  the high temperature part of (27) becomes,

$$\begin{aligned} I(t, u, d-3) &= -\frac{u^2}{16\pi^2} \left(\frac{1}{\epsilon}\right) - \frac{1}{6\pi^2} \zeta(-3) t^2 - \frac{1}{4\pi^2} \zeta(-1) t u + \frac{u^2}{32\pi^2} \ln \frac{u^2}{M^2} \\ &- \frac{1}{12\pi^2} u^{\frac{3}{2}} t^{\frac{1}{2}} + \frac{1}{32\pi^2} u^2 \ln u/t \\ &- \frac{u^2}{16\pi^2} \left( \gamma - 3/4 + \frac{1}{2} \psi(3) - \frac{1}{2} \ln \frac{M^2}{\pi} \right) + O(t^{-1}) \end{aligned} \quad (32)$$

with

$$\psi(x) = \frac{d}{dx} \Gamma[x]$$

Here we have also introduced  $M$  as an arbitrary scale of renormalization. From the above expression we can easily see that there will be imaginary contributions from some of the terms involved. If we closely inspect the possible  $u$ 's from eq. (24) this becomes clear. Apart from the irrelevant constants and after getting rid of the pole term by a suitable counterterm we can write the effective potential at high temperatures as,

$$V_{eff} = V + \frac{1}{64\pi^2} \sum_{i=1} |A_i|^2 \ln \left( \frac{|A_i|}{M^2} \right) + \frac{9\kappa^4 V^2}{64\pi^2} \ln \left( \frac{\kappa^2 V}{M^2} \right) + \text{imaginary terms} + \text{real terms dependent on } T. \quad (33)$$

The imaginary contribution for the high temperature part turns out to be proportional to  $T$  and has the same sign as the zero temperature imaginary piece. Thus the finite temperature contribution reinforces the instability inherent in the imaginary part of the zero temperature effective potential.

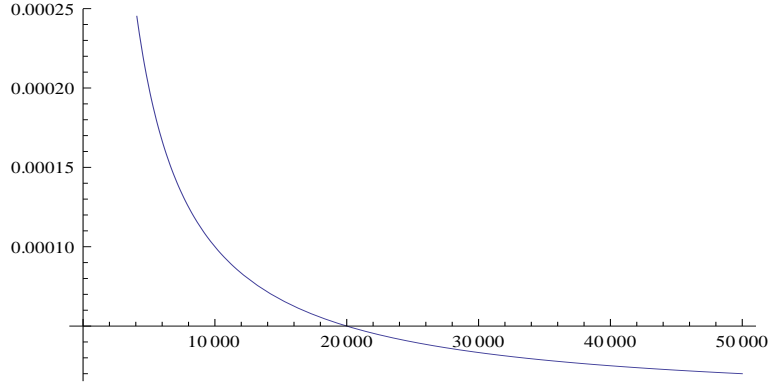


Figure 1: Plot of temperature dependent imaginary part of EP versus  $x$ , for  $x \ll 1$

It is clear that since the dimensional pole term is proportional to  $V^2$  instead of  $V$  the theory is non-renormalizable. However, our primary interest is not the ultraviolet completion of the theory, but rather its infrared instabilities at zero and finite temperature.

To obtain the low temperature limit of eq. (27) we now have to use the following identity

$$\sum_{n=-\infty}^{\infty} e^{-\tau n^2} = \left( \frac{\pi}{t\tau} \right)^{1/2} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2 / t\tau} \quad (34)$$

With the help of this we can write eq. (27) as

$$I(t, u, d) = -\frac{1}{2\pi^{1/2}} \sum_{n=-\infty}^{n=\infty} \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t\tau} \quad (35)$$

This breaks into two parts, one is temperature dependent and other is zero temperature remnant of the effective potential. Once again the pole term is independent of temperature. After separating out the  $n = 0$  piece from the above expression we get,

$$-\frac{1}{2\pi^{1/2}} \frac{1}{(4\pi)^{d/2}} u^{(d+1)/2} \Gamma \left( -\frac{d}{2} - \frac{1}{2} \right) - \frac{1}{\pi^{1/2}} \sum_{n=1}^{n=\infty} \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t\tau} \quad (36)$$

From the first term we easily extract the pole term

$$-\frac{u^2}{16\pi^2} \left( \frac{1}{\epsilon} \right) - \frac{u^2}{32\pi^2} \left( \psi(3) + \ln \frac{4\pi}{u} \right) + O(\epsilon) \quad (37)$$

We can perform the  $\tau$  integration to get the temperature dependent part. The result is given in terms of modified Bessel function [20],

$$\int_0^\infty d\tau \tau^{-(d+3)/2} e^{-\tau t} e^{-\pi^2 n^2 / t \tau} = \left( \frac{tu}{\pi^2 n^2} \right) K_2(2\sqrt{\pi^2 n^2 u/t}) \quad (38)$$

The low-temperature behaviour of the integral eq. (35)

$$I(t, u, 3 - \epsilon) = -\frac{u^2}{16\pi^2} \left( \frac{1}{\epsilon} \right) - \frac{u^2}{32\pi^2} \left( \psi(3) + \ln \frac{4\pi}{u} - \ln \frac{u}{M^2} \right) \quad (39)$$

$$- \frac{u^2}{2^{\frac{1}{2}}} \sum_{n=1}^{\infty} \left( \frac{t}{4\pi^2 n^2 u} \right)^{\frac{5}{4}} e^{(-4\pi^2 n^2 u/t)^{1/2}}, \quad (40)$$

where for large value of the argument we have taken an asymptotic expansion for the modified Bessel function.

To analyze the low temperature behaviour of the potential we ignore any imaginary part coming from the  $A_i$ 's and will only concentrate on  $u = -\kappa^2 V$  part. Then at Low temperature imaginary contribution for effective potential has the following form

$$\frac{\kappa^4 V^2}{32\pi} + \frac{\kappa^4 V^2}{2} \left( \frac{T^2}{\kappa^2 V} \right)^{\frac{5}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}} (\cos nx - \sin nx) \quad (41)$$

where  $x = \frac{\kappa V^{\frac{1}{2}}}{T}$

The second term above is temperature dependent. We can approximate the sum as a integral over  $n$  as  $n$  goes upto infinity or we can compute the sum exactly. Performing both using MATHEMATICA we have found the behaviour of the second term of eq. (41) is oscillatory with damping amplitude for values of  $x > 0.6$  approximately and for large value of  $x$  i.e. for  $T \rightarrow 0$  the oscillations die away and temperature dependent imaginary part vanishes.

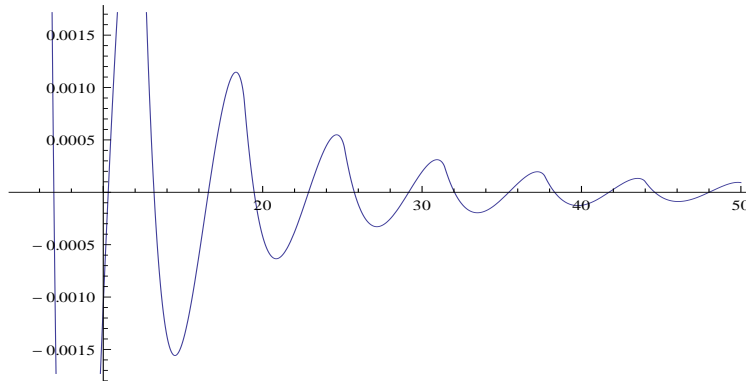


Figure 2: Plot of temperature dependent imaginary part of EP versus  $x$ ,  $x \gg 1$

The analysis above is based on a one-loop effective action evaluated in a certain gauge. However, there is a gauge invariant way of calculating the one-loop effective potential due to Vilkovisky and De-Witt [21, 22]. If one calculates the effective potential in that method there is hardly any qualitative change in the effective potential. However the numbers do change. We state the important factors here without going into details of the calculation scheme but to get unambiguous result it is better to calculate the effective potential in Vilkovisky-DeWitt method.

The structure of the potential is almost the same for VD approach except the prefactor of the momentum integral changes to 5 in place of 9 in eq. (22). The constants  $a$  and  $b$  in this case are

$$a = V'' - \frac{3}{2}\kappa^2 V$$

and

$$b = \frac{1}{2}\kappa^4 V^2 - \kappa^2 V V'' + \frac{3}{4}\kappa^2 V'^2$$

The above analysis may be redone with these minor changes which will not change the qualitative nature of the solution much.

## 5 Conclusion

We have shown that for a theory in which the graviton field is minimally coupled to a Higgs scalar field, flat Minkowski spacetime is unstable. This instability is exhibited as a tachyonic mode in the one-loop propagator. A constant scalar field background resembles a thermal bath which backreacts to the gravitons to produce the instability in the system. This infrared instability in the effective propagator may be regarded as a graviton induced Jeans-like instability. This infrared instability is also manifested in the one-loop effective potential as an imaginary term, independent of the ultraviolet cut off. This term arises also from infrared limit of the loop integrals.

We have also computed the effect of finite temperature for the graviton-Higgs theory and compared it with the zero temperature result. The high temperature sector involves temperature dependent terms which adds to the imaginary contribution obtained in the zero temperature case. Infrared sector exhibits an instability because of imaginary contribution from both zero temperature and temperature dependent part. Moreover it exhibits an oscillatory behaviour which eventually gets damped out as we lower the temperature.

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